

Will It Tile? Try the Conway Criterion!

Doris Schattschneider

In 1975 Martin Gardner described a discovery by John H. Conway that proved very useful in determining which polyominoes (stuck-together squares) might pave the plane. Gardner describes Conway's sufficient condition for a tile to pave the plane as follows:

We examine a given [tile] to see if its perimeter can be divided into six parts, a, b, c, d, e and f , that meet the following requirements: (1) Two opposite edges, a and d , are 'parallel' in the sense that they are congruent and in the same orientation. (2) Each of the other four edges, b, c, e and f , is centrosymmetric; that is, they are unaltered by a 180° rotation around the midpoint. ...Any of the six edges may be empty (nonexistent). [6]

The polyomino tile A in Figure 1 illustrates this. It turns out that this criterion applies even to tiles that are not polyominoes.

In my own investigations of tiling problems, I found this test an excellent one, yet it became apparent that certain tiles could fit this description and still fail to pave the plane. Tile B in Figure 1 is such an example. Clearly what is wrong is that the edges a and d , which are congruent and parallel as required, cannot interlock with each other to fulfill the intention of that requirement. The somewhat ambiguous term "parallel" in Gardner's description seems to be the source of the problem. The use of this term might actually suggest that the endpoints of the edges a and d must form a parallelogram (Gobel's version in [7] actually states this); in fact, paving tiles can have these endpoints collinear. Tile C in Figure 1 is of this type; here the segments a and d are congruent by a *translation* (the usual definition of "parallel" in transformation geometry) along the line containing all four endpoints of a and d .

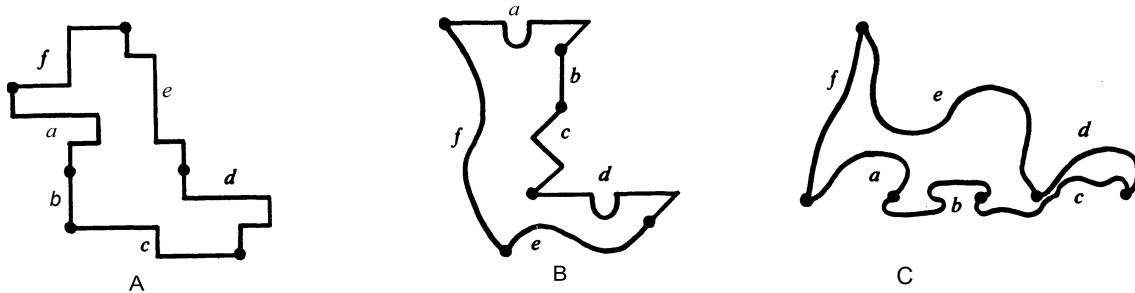


Figure 1

In searching for a reformulation of the Conway Criterion that would remove these ambiguities, and explain how it works, I found that although Conway's test is by no means necessary for a tile to pave the plane, it actually characterizes those tiles that pave the plane "nicely" using only half-turns (180° rotations). Here I present my reformulation of the test, approved by John H. Conway (1937–2020) and focus on why and how the criterion works. (R. Bantegnie states a similar version in [1].)

Throughout, a *tile* (noun) will be a closed topological disk in the plane with its boundary a simple closed curve. To *tile* (verb), or pave the plane with a tile T is to cover the plane with congruent images of T , without gaps or overlaps (except for boundaries), forming a jigsaw puzzle of infinite size called a *tiling by T* . If two points v_i, v_j are on the boundary of a tile, we will denote by $v_i v_j$ that portion of the boundary that connects v_i to v_j and call $v_i v_j$ a *boundary segment*.

Conway Criterion:

A tile T can pave the plane by half-turns if there are six consecutive points $v_1, \dots, v_6$, at least three distinct, on the boundary of T (consecutive in the sense of traveling a cycle around the boundary) that satisfy the following conditions:

- (i) v_1v_2 is congruent to v_5v_4 by a translation τ in which $\tau(v_1) = v_5$ and $\tau(v_2) = v_4$
- (ii) $v_2v_3, v_3v_4, v_5v_6, v_6v_1$ are centrosymmetric.

The v_i are vertices of a tiling by T that is generated by half-turns (180° rotations) about the midpoints of the centrosymmetric segments (and perhaps about some of the v_i).

A tile that satisfies the two conditions of the Conway Criterion will be called a *Conway tile*, and the tiling generated by the specified half-turns its *Conway tiling*. Note that any Conway tile will also satisfy the two conditions in Gardner's description, but by labeling vertices, rather than segments on the boundary, the way in which the "parallel" segments must match is clear. If the vertices (black dots) of the three tiles in Figure 1 are labeled consecutively, beginning with v_1 and v_2 as endpoints of the boundary segment a , it is easily seen that tile B is not a Conway tile, while tiles A and C are.

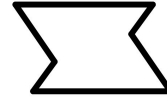
To explain why the criterion is sufficient, we begin by describing arbitrary tiles that pave the plane in the simplest manner. Such tiles are just translated in two directions, laid in parallel rows, with edges matching. The boundary of such tiles must be composed of matching pairs of "opposite" segments that interlock when the tile is translated. There can be only two pairs or three pairs of opposite matching boundary segments if we take the ends of the segments to be vertices of the tiling (i.e., a point in the tiling where at least three tiles meet). Coxeter proves this for convex polygon tiles [4]; Niven gives an even more general proof in [12]. Both proofs remain valid for arbitrary tiles if the term "vertex of a tile" is defined to be a point on its boundary that is a vertex of the associated tiling. Thus, the only tiles that pave the plane by successive translations in two directions, matching their edges, are generalizations of parallelograms or par-hexagons (a convex or nonconvex hexagon with opposite sides equal and parallel). Either a parallelogram tile or a par-hexagon tile can also fill the plane by successive half-turns. This is because these polygons and their edges are centrosymmetric figures.



Parallelogram



Convex par-hexagon



Nonconvex par-hexagon

The following facts are at work [2]:

$(H+H)$: the sum (composite) of two half-turns about points x and y , respectively, is a translation whose vector is $\overrightarrow{2xy}$.

$(T+H)$: a translation τ followed by a half-turn about a point y is equivalent to a half-turn about a point x located so that the vector associated to τ is $\overrightarrow{2xy}$.

The next two observations are illustrated in Figure 2. Property $(H+H)$ implies that a half-turn about a midpoint of an edge of a parallelogram or par-hexagon combined with a half-turn about the center of the tile creates the translation that matches that edge to its opposite parallel edge, and moves the tile to an adjacent tile. Properties $(H+H)$ and $(T+H)$ together imply that all sums of these successive half-turns will yield translations and half-turns that leave the tiling invariant. This is because the half-turn centers that are used to generate the tiling form a parallelogram lattice with edges of the parallelograms parallel to and one-half the length of the translation vectors that generate the tiling.

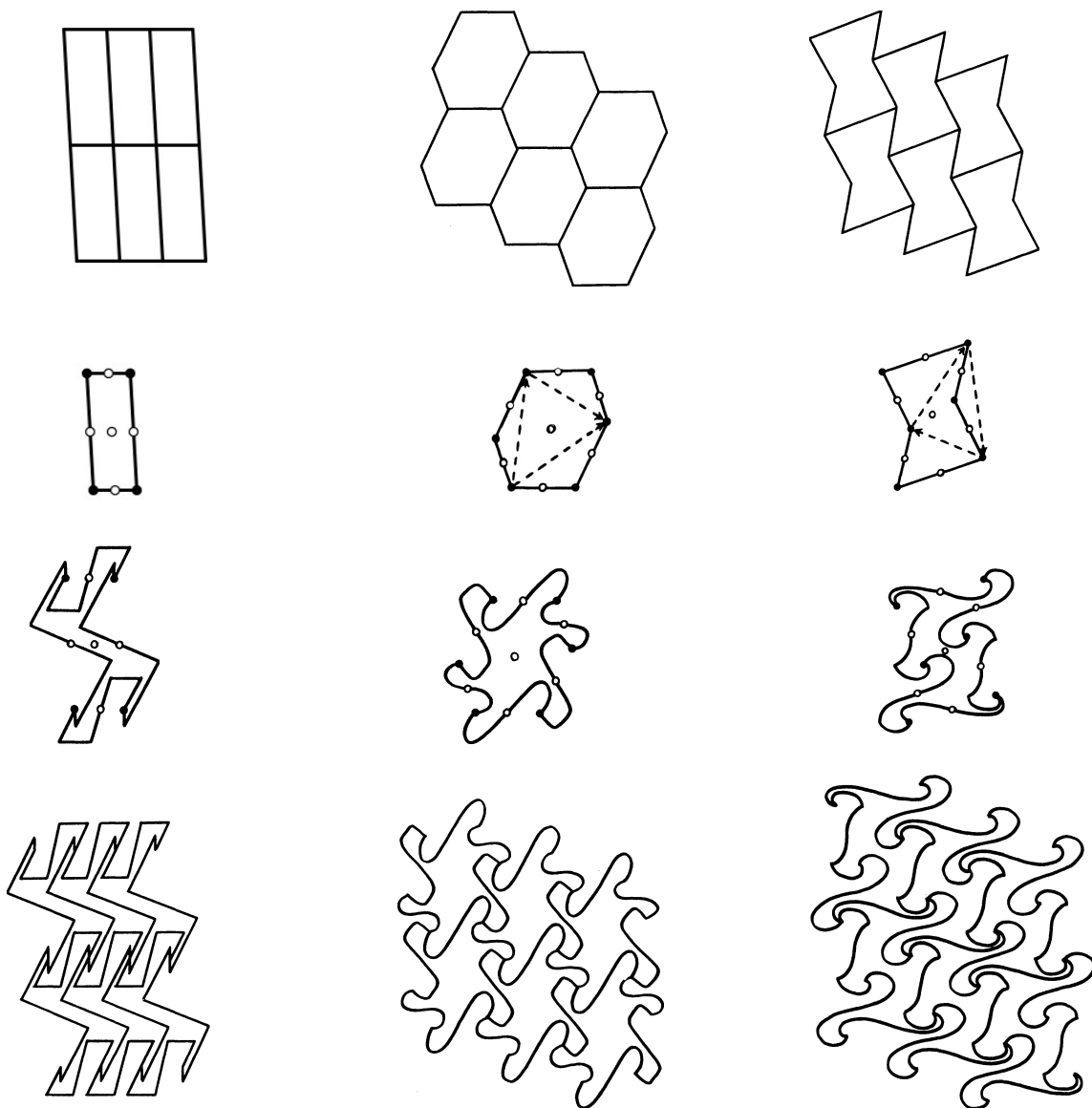


Figure 2. Familiar tilings of the plane by parallelograms and par-hexagons. These polygons have opposite pairs of sides that match by translations (dotted vectors); each of these translations is the sum of two half-turns whose centers are located at midpoints of sides and at the center of the polygon (small circles). Replacing pairs of opposite edges of the polygons with congruent centrosymmetric segments produces $p2$ unit tiles that pave the plane using the same half-turns and translations.

If pairs of opposite edges of a parallelogram or par-hexagon are replaced by pairs of congruent boundary segments that are centrosymmetric, then this crucial placement of half-turn centers is preserved, and so a more general tile is created that can pave the plane by half-turns in the same manner. This description characterizes $p2$ unit tiles. The name " $p2$ " refers to the type of symmetry group (generated by half-turns) that acts transitively on the tiles in the associated tiling [13]; the word "unit" refers to the fact that the translation subgroup of this group also acts transitively on the tiles of this tiling. Figure 3 shows that it is sometimes possible to create a $p2$ unit tile from a par-hexagon whose boundary is not a simple closed curve. You try to choose centrosymmetric boundary segments that join consecutive vertices so as to eliminate intersections.

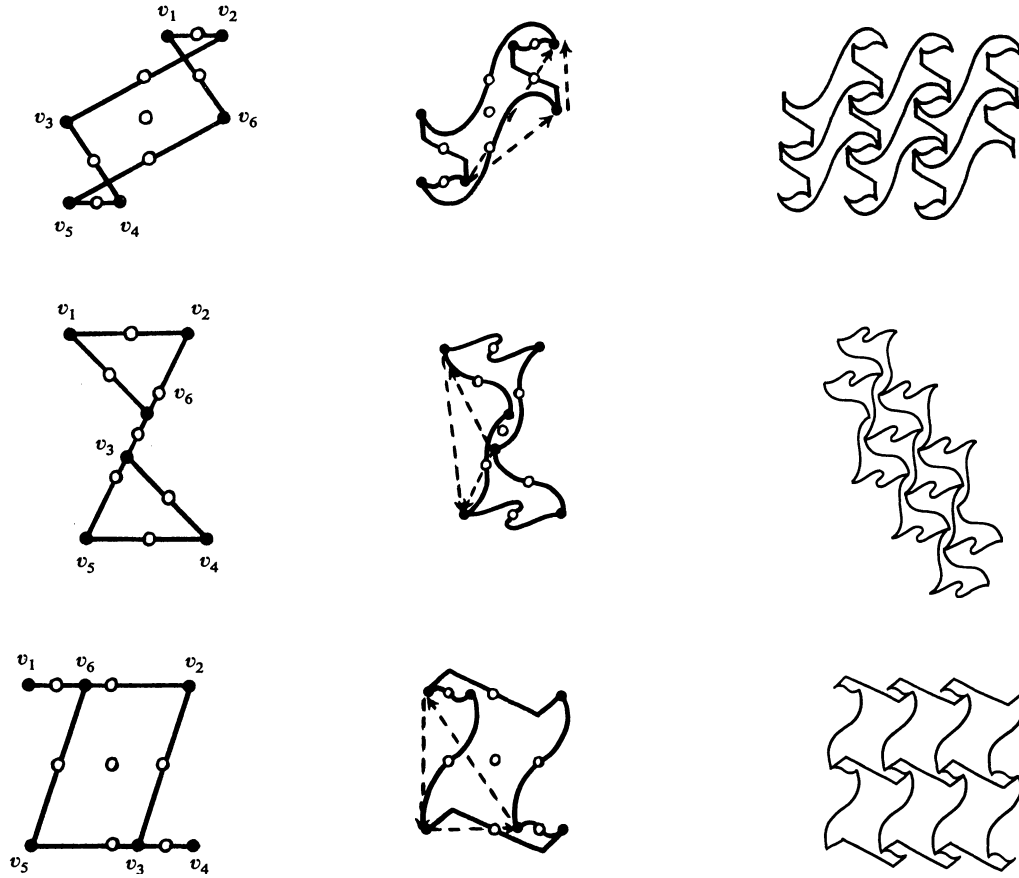


Figure 3. Even a re-entrant or 'degenerate' par-hexagon whose boundary is not a simple closed curve may possibly give rise to a $p2$ unit tile. Vertices of the par-hexagons are labeled consecutively.

We can now show why any Conway tile paves the plane by half-turns as specified in the Conway criterion. If such a tile is a $p2$ unit tile, in which case (i) is satisfied for opposite pairs of boundary segments and (ii) is satisfied by all boundary segments, it certainly tiles in this manner. For any other Conway tile T , it is sufficient to show that a single half-turn about the midpoint of one of the centrosymmetric segments in (ii) creates a $p2$ unit tile $\bar{T}$ whose associated tiling is generated by the half-turns specified in the criterion. By its construction, we know that $\bar{T}$ is a centrosymmetric tile.

If on the one hand T is a Conway tile with $v_1 = v_2$ (and $v_4 = v_5$), then the boundary of T consists of three or four centrosymmetric boundary segments. It seems obvious (see Figure 4) that the new tile $\bar{T}$ is a $p2$ unit tile. In the first case, $\bar{T}$ is a generalized parallelogram and in the second, a generalized par-hexagon.

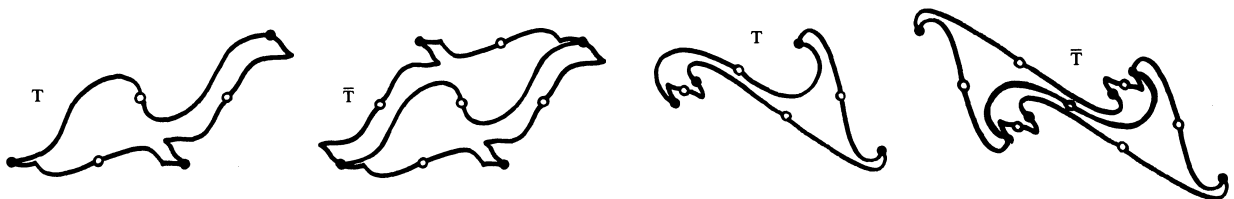


Figure 4

Actually, some elementary facts of geometry guarantee that the midpoints of the centrosymmetric boundary segments of $\bar{T}$ form the correct parallelogram lattice required of a $p2$ unit tile. The following facts are standard geometry exercises::

- (1) If p, q, r are three distinct points in the plane, and m_1, m_2 are the midpoints of the line segments pq and qr , then m_1m_2 is parallel to pr and m_1m_2 is one-half the length of pr .
- (2) If p, q, r, s are four distinct points in the plane, not all collinear, then the midpoints of the line segments pq, qr, rs, sp form a parallelogram.

A proof of (2) for the case when no three of the four points are collinear appears in [3]. Note that both (1) and (2) remain true if the phrase "midpoints of the line segments" is replaced by "midpoints of the centrosymmetric boundary segments," since these midpoints are, in fact, the same points. Although the parallelogram of midpoints in (2) may be degenerate, collapsed on a line segment, this cannot occur in our case since the points p, q, r, s are the vertices v_i on a simple closed curve, namely, the boundary of T . Thus in both cases T indeed tiles by half-turns about the midpoints of its centrosymmetric boundary segments.

If on the other hand T is a Conway tile with $v_1 \neq v_2$ (and $v_4 \neq v_5$), then the segments v_1v_2 and v_5v_4 are congruent by the translation τ , and two, three or four centrosymmetric boundary segments complete the boundary of T . Relabeling vertices if necessary, we may assume that we have chosen the midpoint m of v_2v_3 about which to half-turn T to create $\bar{T}$. To prove $\bar{T}$ is the desired $p2$ unit tile, four different cases must be considered, determined by the number and location of distinct vertices of T (relative to v_2v_3). We prove one case below (Figure 5) and leave the remaining similar arguments to the reader (Figure 6).

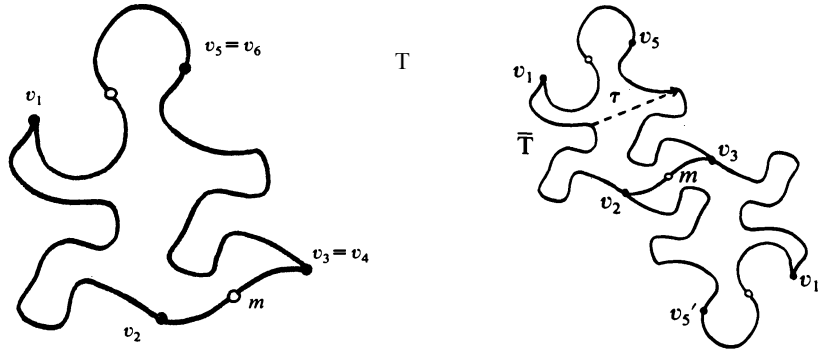


Figure 5

When T has two centrosymmetric boundary segments that join v_1v_2 and v_4v_5 , then $v_3=v_4$ and we can label $v_5=v_6$. Denote by h_m the half-turn about m , and let v_1' and v_5' be the images of v_1 and v_5 by h_m . Since $\tau(v_2) = v_4 = v_3$, property $(T+H)$ implies that the sum of τ followed by h_m is a half-turn about v_2 . This shows that the boundary segment $v_1v_2v_5'$ of $\bar{T}$ is centrosymmetric about v_2 . Since $\bar{T}$ is centrosymmetric, and has two pairs of "opposite" centrosymmetric boundary segments (namely v_1v_5 and $v_5'v_1'$ and $v_1v_2v_5'$ and $v_5v_3v_1'$), $\bar{T}$ is a $p2$ unit tile that paves by half-turns about the midpoints of v_1v_5 , v_2v_3 , and the vertex v_2 of T .

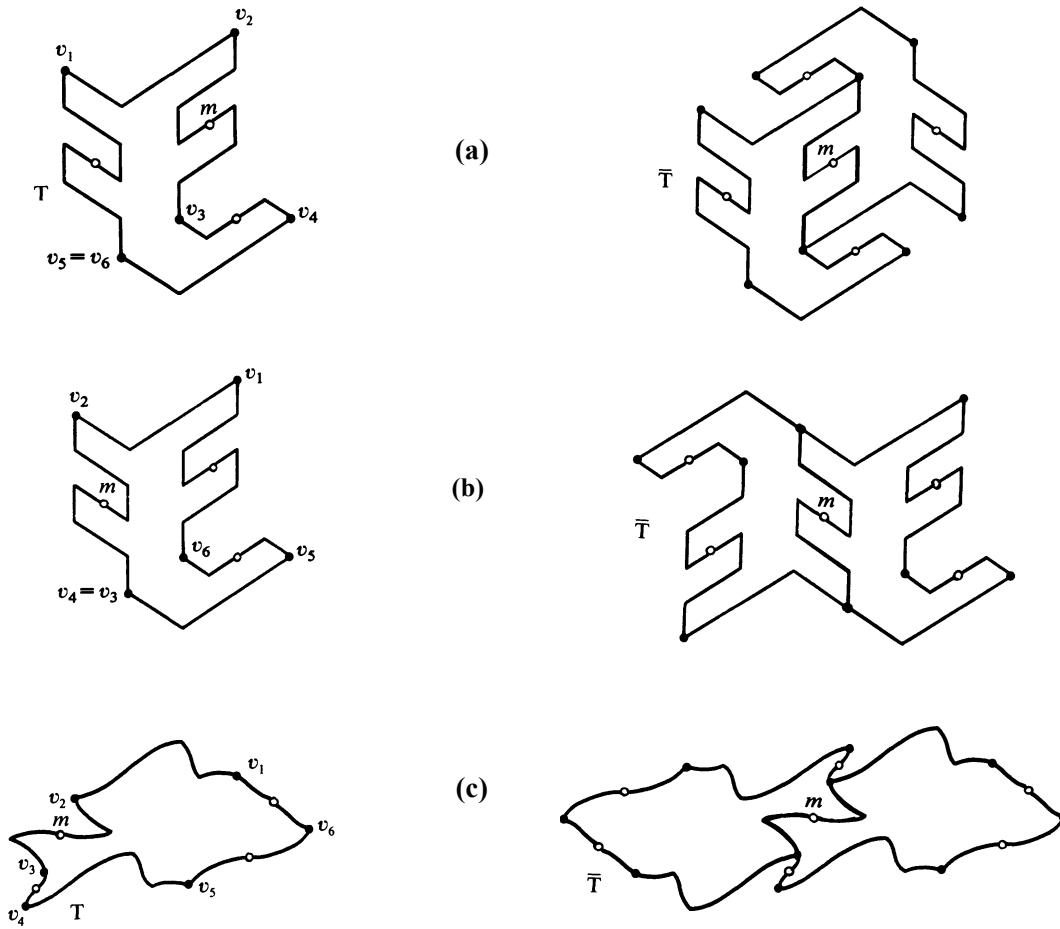


Figure 6. (a) Tile with $v_5 = v_6$. (b) Tile with $v_3 = v_4$. (c) Tile with six distinct vertices.

We now turn to the converse and ask which tiles must be Conway tiles? Since a Conway tile T or its double tile $\bar{T}$ is a $p2$ unit tile, our earlier remarks show that the half-turns that generate its Conway tiling leave that tiling invariant. It is also possible for a tile to pave by half-turns in a "sloppy" way. For example, the tiling by triangles in Figure 7 can be continued (randomly shifting rows) to insure that although each triangle half-turns into each of its neighbors, none of these half-turns leaves the tiling invariant.

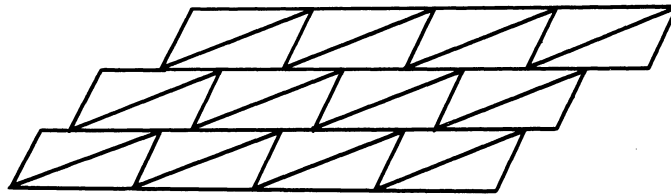


Figure 7

The Conway Criterion actually characterizes those tiles that pave "nicely" by half-turns:

If a tile T paves the plane by half-turns that leave the tiling invariant, then T is a Conway tile.

If a tile paves by such half-turns, then these motions generate a subgroup of the symmetry group of the tiling that acts transitively on the tiles. Tiles that have $p2$ isohedral tilings (a $p2$

symmetry group acts transitively on the tiles) are shown in [8] and [9]; the results there verify our statement above. Since details are missing in these references, we offer a constructive proof that also shows how, by hand or by using computer graphics equipment, $p2$ isohedral tilings can be created.

If T is a tile that paves by half-turns, then the centers of these half-turns must be located on the boundary of T (a point about which T half-turns into an adjacent copy of T) or at the "center" of T (a point about which T half-turns onto itself). If the half-turns must leave the tiling invariant, then three half-turns about noncollinear centers will generate the whole tiling; these centers are carried into a lattice of half-turn centers for the whole tiling. To show that T must be a Conway tile, we create tilings by such a tile T , then examine all of the "distinct" ways in which this can be done.

Beginning with three non-collinear half-turn centers, properties $(H + H)$ and $(T + H)$ imply that the sum of these three half-turns creates a fourth half-turn whose center completes a parallelogram of half-turn centers. Joining three or all of these centers with curves creates connected segments that, when acted on by the group G generated by these half-turns, form a planar network enclosing tiles. We will call these arbitrary curves chosen to connect the half-turn centers *generating boundary segments*. In choosing the generating boundary segments, care must be taken so that the resulting tiling will be one for which the group G acts transitively on the tiles. In this way, we create a tiling by a single tile T that paves the plane "nicely" by half-turns. A little thoughtful experimentation easily yields all the distinct ways in which generating boundary segments can connect the half-turn centers in the desired manner. If we classify generating boundary segments according to the location of vertices of the associated tiling (relative to the half-turn centers) and the valences of these vertices, then there are just seven distinct types (Figure 8). The choice for the shapes of the generating boundary segments is limitless. All of the types of $p2$ isohedral tilings listed in [8, 9] are accounted for in this scheme and, in addition, it is clear that each of the 7 types of tiles created satisfies the Conway Criterion.

In Figure 8 (next page) we have chosen to have the generating half-turn centers (small circles) form the same parallelogram for all seven cases. In addition, the first four tiles share some of the same boundary segments, and the last three tiles share some of the same boundary segments. This illustrates how the different types are related. Black dots on the generating boundary segments become vertices of the tile T and its tiling; all vertices of T and of its associated tiling are images of these by elements in the group G .

We close with some remarks on the nature of the Conway Criterion and an open question. Why is the test so useful? Its power in deciding if stuck-together squares, triangles and hexagons might tile is not too surprising, since boundaries of such tiles have a high probability of containing centrosymmetric segments. But it is a good first test to try on any tile since 10 of the 17 plane symmetry groups contain a subgroup generated by half-turns [13]. In analyzing the boundary of a tile, you look for centrosymmetric curves; those that are not line segments have characteristic zig-zag, stairstep, or S shapes. Figure 9 shows that there may be several distinct ways of choosing the points v_i on the boundary of a tile to meet the Conway Criterion and each choice produces a different Conway tiling. If a tile T fails the test, T may pave the plane with no half-turns, or a block of copies of T may form a Conway tile. Gardner [6] discusses this, and [9], [10], and [11] provide useful diagrams and formulations of other criteria that suffice for a tile to pave the plane.

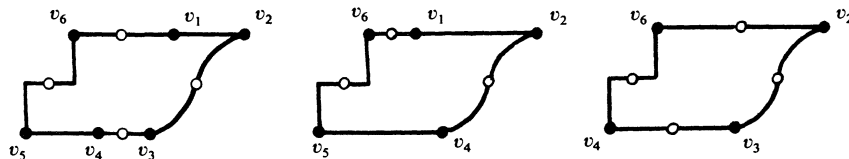


Figure 9

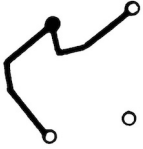
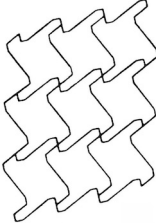
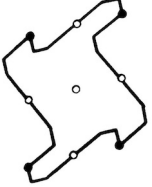
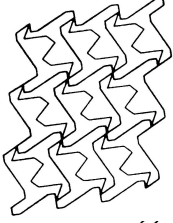
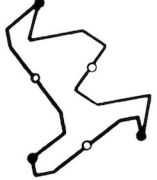
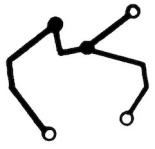
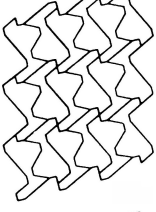
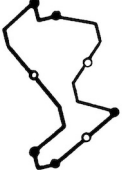
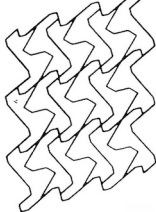
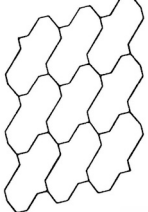
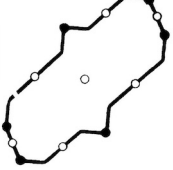
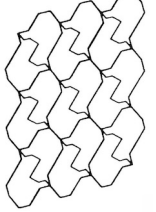
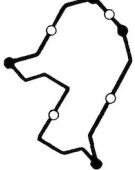
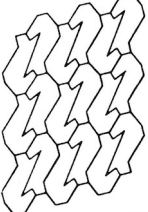
Generating boundary segments	Number, location, and valence of vertices	Associated tiling Isohedral type	Conway tile Heesch type
	1 vertex, at half-turn center, valence 4	IH 57 	<i>p</i> 2 unit tile 
	1 vertex, at half-turn center, valence 6	IH 84 	CCC 
	2 vertices, one at half-turn center, of valence 4, one not at half-turn center, of valence 3	IH 23 	TCTCC 
	2 vertices, both at half-turn centers, of valence 4	IH 47 	TCTC 
	one vertex, not at half-turn center, of valence 3	IH 8 	<i>p</i> 2 unit tile 
	1 vertex, not at half-turn center, of valence 4	IH46 	CCCC 
	2 vertices, both not at half-turn centers, both of valence 3	IH 4 	TCCTCC 

Figure 8. Creation of the seven distinct types of *p*2 isohedral tilings. The isohedral type is that of Grünbaum and Shephard [8,9], and the Heesch type is that of Heesch and Kienzle [10], and is a fundamental domain of the tiling.

Creating tiles using the Conway Criterion can be great fun. Since all triangles and quadrilaterals are Conway tiles, as are some pentagons and hexagons, new Conway tiles can be created from them by altering their sides. Various techniques are explained in [5]. P.A. MacMahon [11] also indicates many kinds of alterations of polygons, preserving the symmetries essential for tiling. You are not limited to just altering polygon tiles. You can make shapes of (three to six) boundary segments and connect them (creating vertices) to make a Conway tile. If you make some of the centrosymmetric boundary segments of a Conway tile congruent, then the tile and its associated tiling may have extra reflection or rotation symmetry. The tiles shown in Figure 10 have associated tilings of type pmg , $cm\bar{m}$, and $p4$, respectively (see [13]).

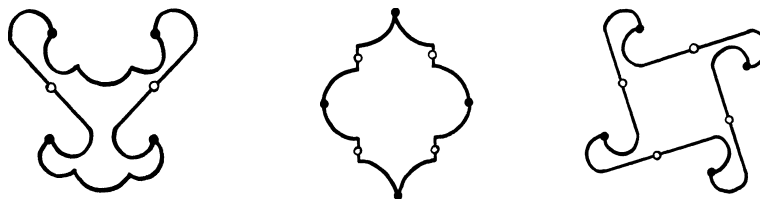


Figure 10

The Conway Criterion, and all other known criteria for testing whether or not a given tile paves the plane, is limited to tiles for which there is an associated periodic tiling (i.e., a group of translations acts transitively on some finite block of the tiling). There seems to be no hope for a test that decides if a given tile paves in any (haphazard) way at all, but perhaps the following question suggested by R. Bantegnie can be answered. If a tile T paves the plane by an application of successive half-turns (which need not be in the symmetry group of the finished tiling), must T be a Conway tile? If not, must there exist a block of copies of T that is a Conway tile?

References

- [1] R. Bantegnie, “Etalements Cristallographiques,” *Acta Math. Acad. Sci. Hungar.*, 30 (1977) 283–302.
- [2] H.S.M. Coxeter, *Introduction to Geometry*, 2nd ed., Wiley, New York, 1969, pp. 41–42.
- [3] H.S.M. Coxeter and S.L. Greitzer, *Geometry Revisited*, New Mathematical Library, no. 19, Mathematical Association of America, 1967, p. 53.
- [4] H.S.M. Coxeter, *Twelve Geometric Essays*, Southern Illinois University Press, 1968, pp. 56–57.
- [5] R. Fathauer, *Tessellations: Mathematics, Art, and Recreation*, A.K. Peters/CRC Press, 2020.
- [6] M. Gardner, “More about tiling the plane: the possibilities of polyominoes, polyiamonds, and polyhexes,” *Scientific American* (August 1975) 112–115.
- [7] F. Gobel, *Geometric Packing and Covering Problems, Packing and Covering In Combinatorics*, A. Schrijver (Editor), Mathematical Centre, Amsterdam, 1979, pp. 179–199.
- [8] B. Grünbaum and G.C. Shephard, “The eighty-one types of isohedral tilings in the plane,” *Mathematical Proceedings of the Cambridge Philosophical Society*, 82 (1977) 177–196.
- [9] B. Grünbaum and G.C. Shephard, *Tilings and Patterns*, W.H. Freeman, 1987; 2nd edition Dover 2016.
- [10] H. Heesch and O. Kienzle, *Flächenschluss*, Springer-Verlag, Berlin, 1963.
- [11] P.A. MacMahon, *New Mathematical Pastimes*, Cambridge University Press, 1930, p. 80ff. Tarquin Reprints 2004, Scholar Select 2015.
- [12] I. Niven, “Convex polygons that cannot tile the plane,” *American Mathematical Monthly* (December 1978) 785–792.
- [13] D. Schattschneider, “The plane symmetry groups: their recognition and notation,” *American Mathematical Monthly* (June–July 1978) 439–450.